PHY 101

by

Dr O. E. **Abe** & J. C. **EHIABHILI**

Course Content

Space and Time

frame of reference

Invariance of physical law

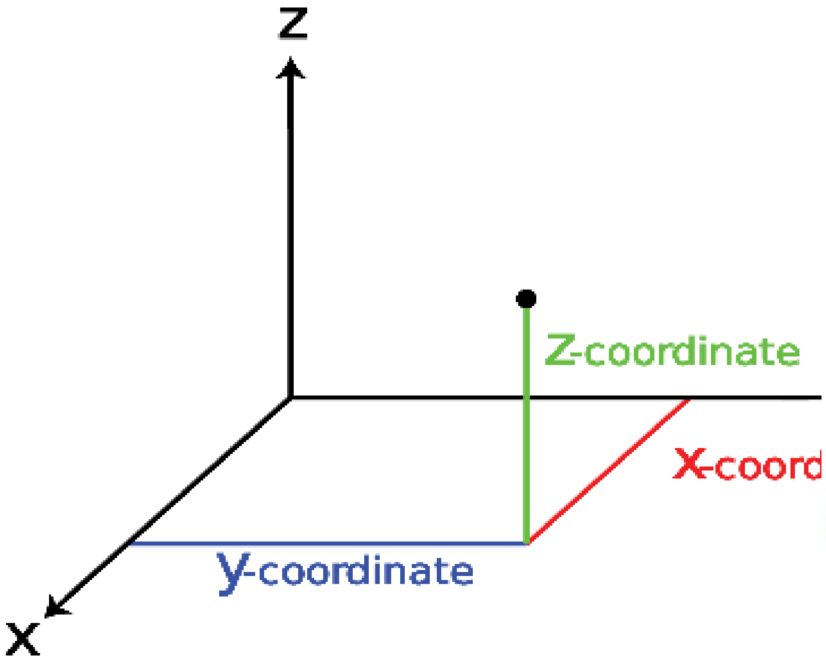
relativity of simultaneity

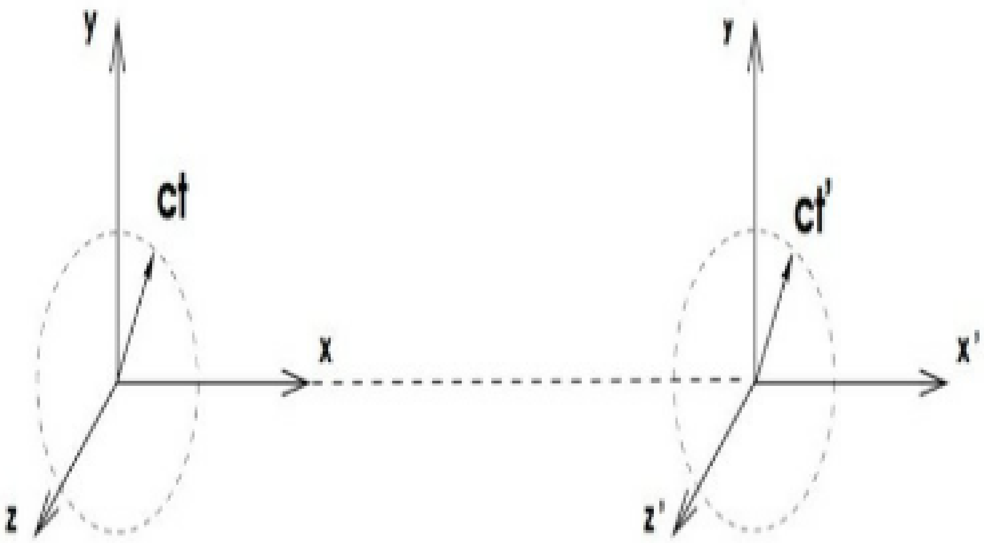
Galilean Invariancee

Space and Time

- In Physics or physical sciences, space is referred to the three-dimensional coordinates points. In other word, it is an arrangement of all possible point locations where logical statement(s) can be derived to solve a specific problem
- Time is one-dimensional quantity that is independent of space. It could be regarded as an origin from which to take duration past or future to any other time instant.

- In classical mechanics, time is separated from space,
- however, in special relativity; time and space are fused together making four-dimensional geometry called space-time





The principle of relativity

The principles of relativity are described into two postulates.

The speed of light in vacuum is always the same; and

It is not possible to measure the absolute speed of an object but only relative can be measured.

The second postulate needs some explanation in the sense that the earth is moving in a rotational motion about its axis round the sun with a known speed, and all the object therein are moving

We can only state how fast one object is moving relative to another.

The second postulate can be expressed as the basic laws of nature,

which are in all reference fames moving with constant velocity relative to each other

are the same in all inertial frame of reference.

A reference frame is any coordinate system relative to which measurements are taken. For example, the position of a bed, chair, and table can be described to the walls of a room. The room is then the reference frame. In another way, if a fly is sitting on a window in a moving car. The window of the car can describe the position of the fly on the car.

An inertia reference frame is a coordinate system in which the law of inertia applies: a body remains at rest unless an unbalance force on it causes it to be accelerated. All reference systems moving with constant velocity relative to distant stars are inertia

The above postulates can be summarised as follow.

The relativistic mass of any object can be describes as $m_R = \gamma m$

where $\gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$ and $\sqrt{1 - {v^2}/{c^2}}$ *is the relativistic factor*

Relativistic Linear Momentum (P) = $P = M_R V = \gamma m V = \frac{mV}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$

Apparently, if you are in motion relative to two points that are fixed distance apart, the distance between the two points appears shorter than if you were at rest relative to them. The ratio of the two points is the relativistic factor.

Relativistic Energy (E) = γmc^2

The Total Energy (E) is the sum of its kinetic and potential (rest) energies

$$E = Kinetic\ Energy + Rest\ Energy$$

$$i.e\ \gamma mc^2 = KE + E_o$$

When a body is at rest $\gamma = 1, KE = 0$

$$\therefore\ E_o = mc^2$$

$$\Longrightarrow KE = \gamma mc^2 - mc^2$$

However, $if\ the\ speed\ of\ the\ object\ is\ much\ less\ than\ speed\ of\ light\ (v \ll c)$

$$KE = \frac{1}{2}mv^2$$

$$expressing\ Energy\ in\ terms\ of\ momentum\ E^2 = m^2c^4 + p^2c^2$$

Relativistic time $Tm = Ts\sqrt{1 - {v^2}/{c^2}}$ where Tm is the moving time and Ts is the proper time.

Relativistic length $Lm = Ls\sqrt{1 - {v^2}/{c^2}}$ where Lm is the moving length and Ls is the proper length. If an object and observer are in relative motion with speed v, then the observer will measure the object as having contracted along the line of motion. The contraction factor is $\sqrt{1 - {v^2}/{c^2}}$. This occurs only along the line of motion. No such contraction is observed perpendicular to the direction of motion. The length of an object measured by an observer at rest relative to it is known as proper length.

Worked Example

(1) Determine the energy required to give an electron a speed equal to 0.9 that of light, starting from rest.

Total energy (E) = KE + PE, $KE = \gamma mc^2 - mc^2$

$$KE = mc^2(\gamma - 1)$$

$$KE = mc^2\left(\frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} - 1\right)$$

$$KE = (9.11\times10^{-31}kg)\times\left(2.998\times\frac{10^8m}{s}\right)\left(\frac{1}{\sqrt{1-(0.9)^2}} - 1\right)$$

$$KE = 1.06\times10^{-13}J$$

(2) Show that $KE\ (\gamma m - m)c^2$ reduces to ${}^1/_2\, mv^2$ when $v \ll c$

$$KE = (\gamma m - m)c^2$$

$$recall\ that\ \gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

$$\therefore KE = \left(\frac{m}{\sqrt{1-\left(\frac{v}{c}\right)^2}} - m\right)c^2$$

$$= mc^2\left(\left(1-\frac{v^2}{c^2}\right)^{-1/2} - 1\right)$$

Let $-\frac{v^2}{c^2}$ be x

$$KE = mc^2\left((1+x)^{-\frac{1}{2}} - 1\right)$$

Expand $(1+x)^{-\frac{1}{2}}$ by binomial expression, we have

$$\therefore\ (1+x)^{-\frac{1}{2}} = 1 + \left(-\tfrac{1}{2}\right)x + \frac{(-1/2)(-3/2)}{2!}x^2 + \cdots$$

Putting the values of x

$$\therefore\ (1+x)^{-\frac{1}{2}} = 1 + \left(\tfrac{1}{2}\right)\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \cdots$$

$$\Rightarrow KE = mc^2\left(\left(1+\left(\frac{1}{2}\right)\frac{v^2}{c^2}+\frac{3}{8}\frac{v^4}{c^4}+\cdots\right)-1\right)$$

$$\therefore\ KE = \frac{1}{2}mv^2 + \frac{3}{8}mv^2\ \frac{v^2}{c^2}+\cdots$$

if v is very smaller than c, all the terms containing c are negligible except the term without c

$$\therefore\ KE = \frac{1}{2}mv^2$$

Assignment 6

(1) A particle is travelling at a speed v such that $^{v}/_{c}$ is 0.99, later the particle decelerated to half speed of light. Find γ for the particle at instance speed.

(2) Determine the speed of an electron having a kinetic energy of $1.6\times10^{-14}J$

Invariance of physical law

The notion of invariance is important in physics, both invariance of physical quantities or properties, and the invariance of equations relating such quantities.

A fundamental example of an invariant quantity, in all forms of relativity, is an event in space-time. To be explicit, suppose two events, E_1, E_2, have the same space-time coordinates in a particular inertial frame of reference,

$$E_1 E_2 \rightarrow \{t_1, x_1, y_1, z_1\} \rightarrow \{t_2, x_2, y_2, z_2\}$$

such that

$$t_1, x_1, y_1, z_1 = t_2 = x_2 = y_2 = z_2$$

then they will have the same space-time coordinates in **every** inertial frame of reference.

That is, if

$$E_1 E_2 \rightarrow \{t'_1, x'_1, y'_1, z'_1\} \rightarrow \{t'_2, x'_2, y'_2, z'_2\}$$

then

$$t'_1, x'_1, y'_1, z'_1 = t'_2 = x'_2 = y'_2 = z'_2$$

Simultaneity

In Galilean relativity, time is invariant, and so we can consider simultaneous events, which are events that occur at the same time, but not necessarily at the same location. Then, if two events are simultaneous in one frame, they will be simultaneous in any other frame.

This, along with time-invariance itself, gets dropped in special relativity.

Spatial separation

Following directly from the invariance of simultaneity, consider the spatial separation between two

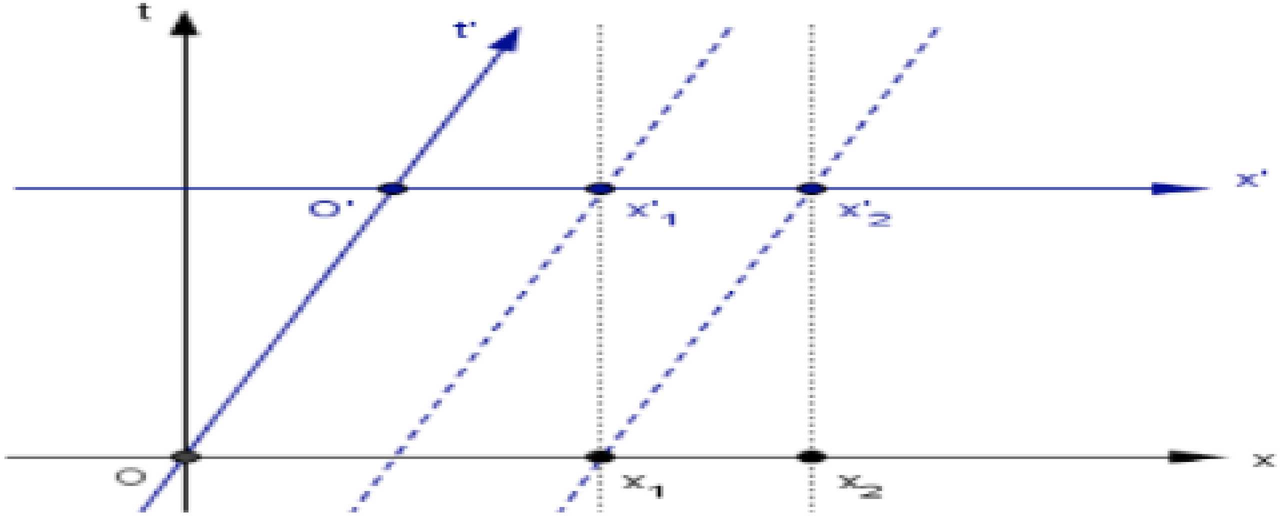


Figure 1.5 - Spatial separation

In figure 1.5, at time $t=t'$, the ends of the stick are at $(t, x_1), (t, x_2)$ in the stationary frame and at $(t', x'_1), (t', x'_2)$ in the moving frame (which is at rest relative to the stick).

The spatial separation as measured in the moving frame is then

$$x'_2 - x'_1 = (x_2 - vt) - (x_1 - vt) = x_2 - vt - x_1 + vt = x_2 - x_1$$

which is the spatial separation as measured in the stationary frame.

Hence, in Galilean relativity, spatial separation is invariant.

Acceleration

Suppose a particle has a trajectory in space (we can stay in one dimension for simplicity), and the coordinates of the trajectory are $x(t)$ in one frame and $x'(t)$ in another (where $t'=t$, of course), moving at uniform speed v relative to the first.

Now, the *position* of the particle is *not* an invariant quantity, since

$$x'(t) = x(t) - vt$$

And neither is the *velocity* of the particle, since

$$\frac{dx'}{dt} = \frac{dx}{dt} - v$$

But **acceleration** is invariant, since

$$\frac{d^2x'}{dt^2} = \frac{d^2x}{dt^2}$$

Mass

Given the fact that position and velocity are not invariant, and that acceleration is invariant, it would make sense to formulate the laws of physics in terms of acceleration.

Newton expressed this explicitly in his force law as $F=ma$. Suppose we have two particles (on the x-axis for simplicity) exerting forces on each other. We always assume that fundamental forces are conservative forces, meaning that they depend only on the distance between the particles. Since we know already that distance - spatial separation - is invariant in Galilean relativity, then the forces acting on the particles are also invariant.

Writing Newton's law as $m = \frac{F}{a}$ we see that mass is also invariant since it is a quotient of invariants.

Finally, we note that the law itself is invariant. That is, F=*ma* holds in all reference frames

In order to compare the laws of physics in two different reference frames, we need to consider transformations of properties between the two frames, and in particular, transformations of coordinates. For now, we shall restrict our attention to motion only along the x- axis and assume time is universal, meaning that the clocks in both frames tick off time at the same - universal - rate.

GALILEAN INVARIANCE

Galilean invariance or **Galilean relativity** states that *the laws of motion are the same in all inertial frames of reference.*

It is a reference frame in relative motion to absolute space. In such a frame, the object does not accelerate.

Galilee Galileo first described this principle using the example of a ship travelling at constant velocity, without rocking, on a smooth sea; any observer performing experiments below the deck would not be able to tell whether the ship was moving or stationary.

- Among the axioms from Newton's theory are:

There exists an absolute space in which Newton's laws are true.

Galilean Transformation Equations

Suppose S’ is proceeding relative to S at a speed v along the x-axis:

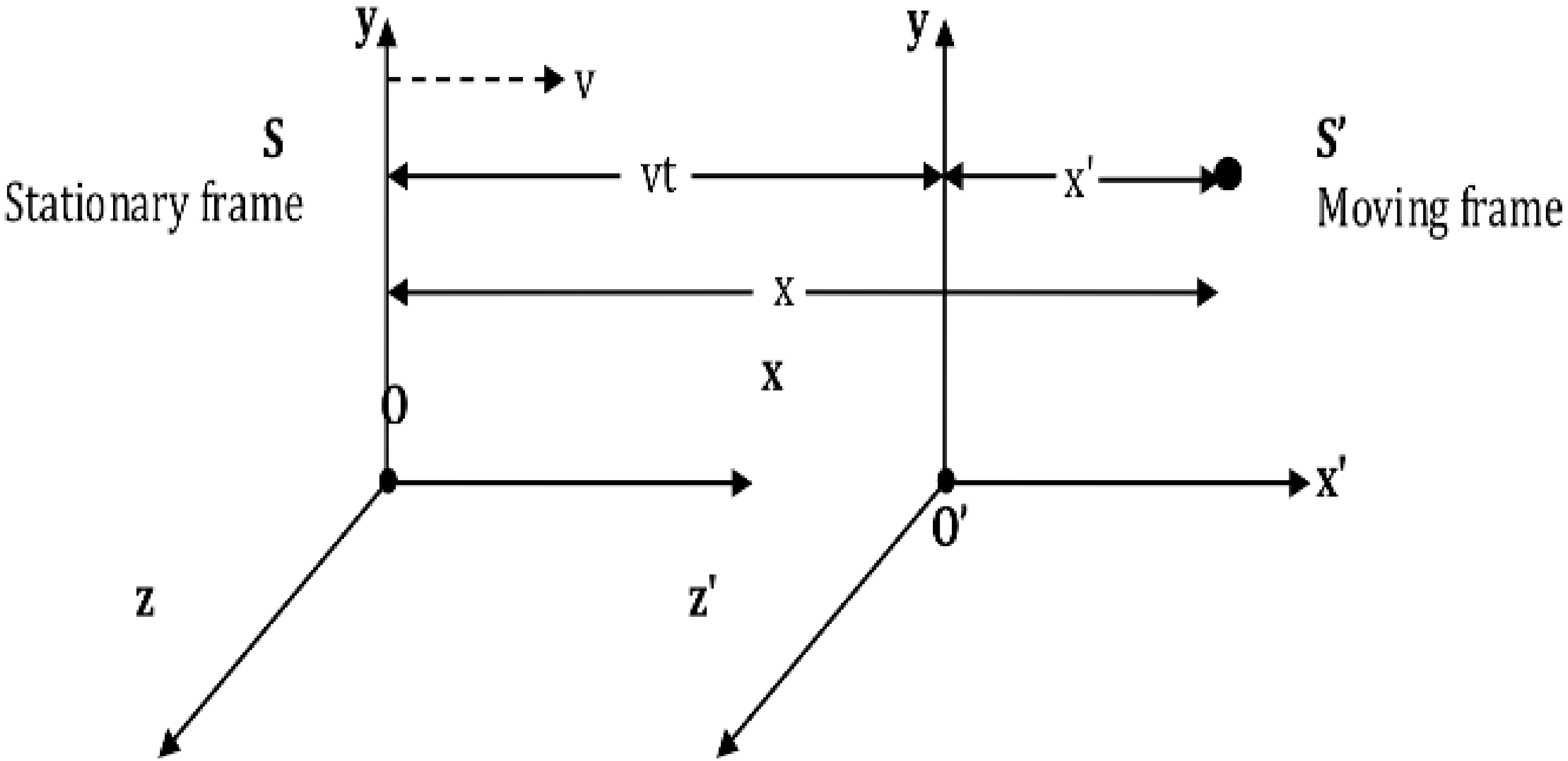


Figure 2: S and S’ inertial frame of reference

Galilean Transformation Equations

We see from the diagram above that $x = x' + vt$. Also, $y = y'$ and $z = z'$ since there are no movements in these directions and $t = t'$ since time is universal. Thus, (x, y, z, t) in S frame corresponds to (x', y', z', t') in S' frame, where:

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}$$

Galilean Transformation
Equations

That is how position transforms and is known as Galilean Transformations

Galilean Velocity Transformation

The velocity u'_x in S' frame in the x' direction as measured by an observer is given as:

$$u'_x = \frac{dx'}{dt'} = \frac{dx'}{dt} \qquad \text{(since t' = t)}$$

$$u'_x = \frac{d}{dt}(x - vt) = \frac{dx}{dt} - v$$

Therefore, $u'_x = u_x - v$ (Galilean velocity transformation)

Galilean Acceleration Transformation

How does acceleration transform?

$$a'_x = \frac{du'_x}{dt'} = \frac{du'_x}{dt}$$

$$a'_x = \frac{d}{dt}(u_x - v) = \frac{du_x}{dt} - \frac{dv}{dt}$$

Since v is constant, $\frac{dv}{dt} = 0$

Therefore, $a'_x = a_x$

This means that the acceleration is the same in both frames. Thus, we say that acceleration in both frames is Galilean Invariant. Velocity is not invariant because it differs in both frames by a constant factor i.e. the relative velocity of both frames.

We can also show that the law of conservation of momentum (Newton's second law of motion) F = ma, is invariant under Galilean transformation.

In the S' frame of reference, $F' = ma'_x$

$$F' = m\frac{du'_x}{dt} = m\frac{d}{dt}\left[\frac{dx'}{dt}\right]$$

$$F' = m\frac{d}{dt}\left[\frac{d}{dt}(x - vt)\right] = m\frac{d}{dt}\left[\frac{dx}{dt} - v\right]$$

$$F' = m\frac{d}{dt}[u_x - v]$$

$$F' = m\left[\frac{du_x}{dt} - \frac{dv}{dt}\right]$$

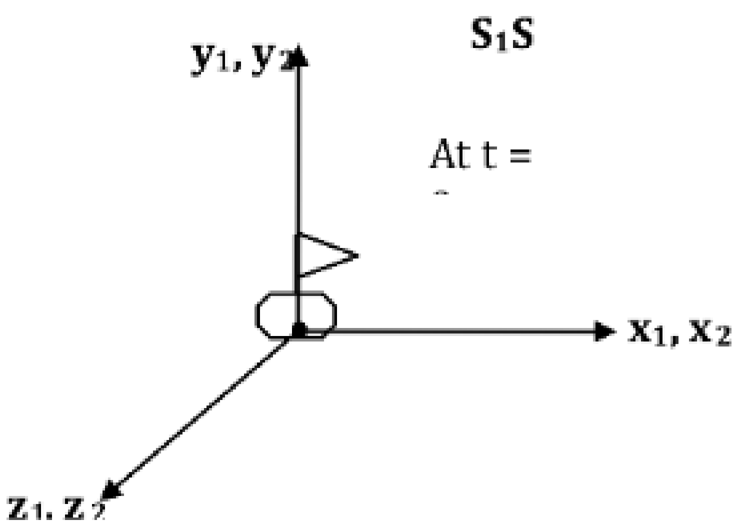
But $dv/_{dt} = 0$

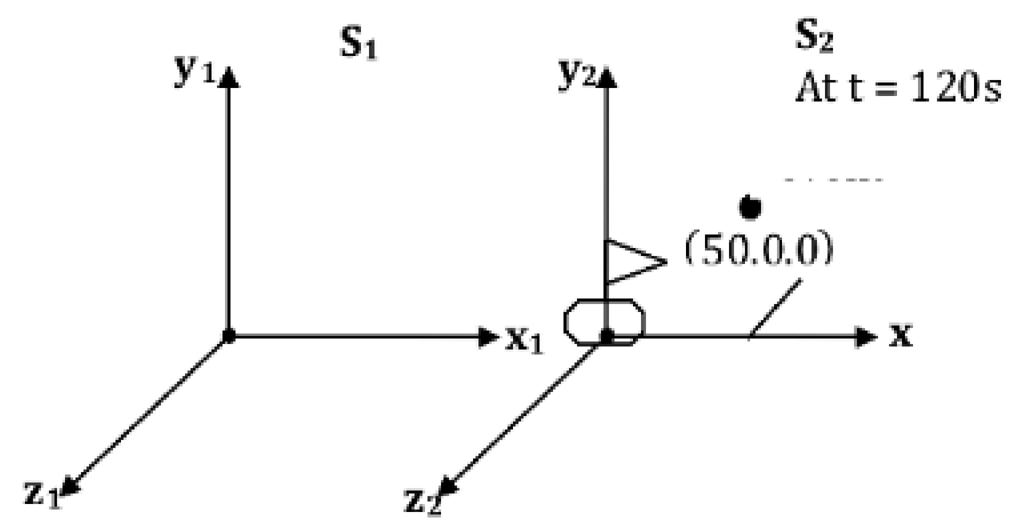
Therefore, $F' = ma_x = F$

Examples

1: A boat that is initially stationary begins to move with a velocity of 10m/s with respect to $x -$ axis in the positive direction. Let the water on which it moves be reference frame 1 and the boat represents reference frame 2. Assume that reference frame 1 is always stationary and at $t = 0$ seconds, the two frames coincide. If we are in reference frame 2 and an event takes place by the boat at point ($x_2 = 50$, $y_2 = 0$, $z_2 = 0$) after the boat travelled for 120s, find the equivalent

Solution





Initially, the origin of the two frames (S_1S_2) coincide at t = 0 seconds

Since the boat is travelling in the positive x direction,

$$x_1 = x_2 + vt, \;\; y_1 = y_2, \;\; z_1 = z_2 \; and \; t_1 = t_2$$

However, at t = 120s, $x_2 = 50$, $y_2 = 0$, $z_2 = 0$

Therefore, $x_1 = x_2 + vt$

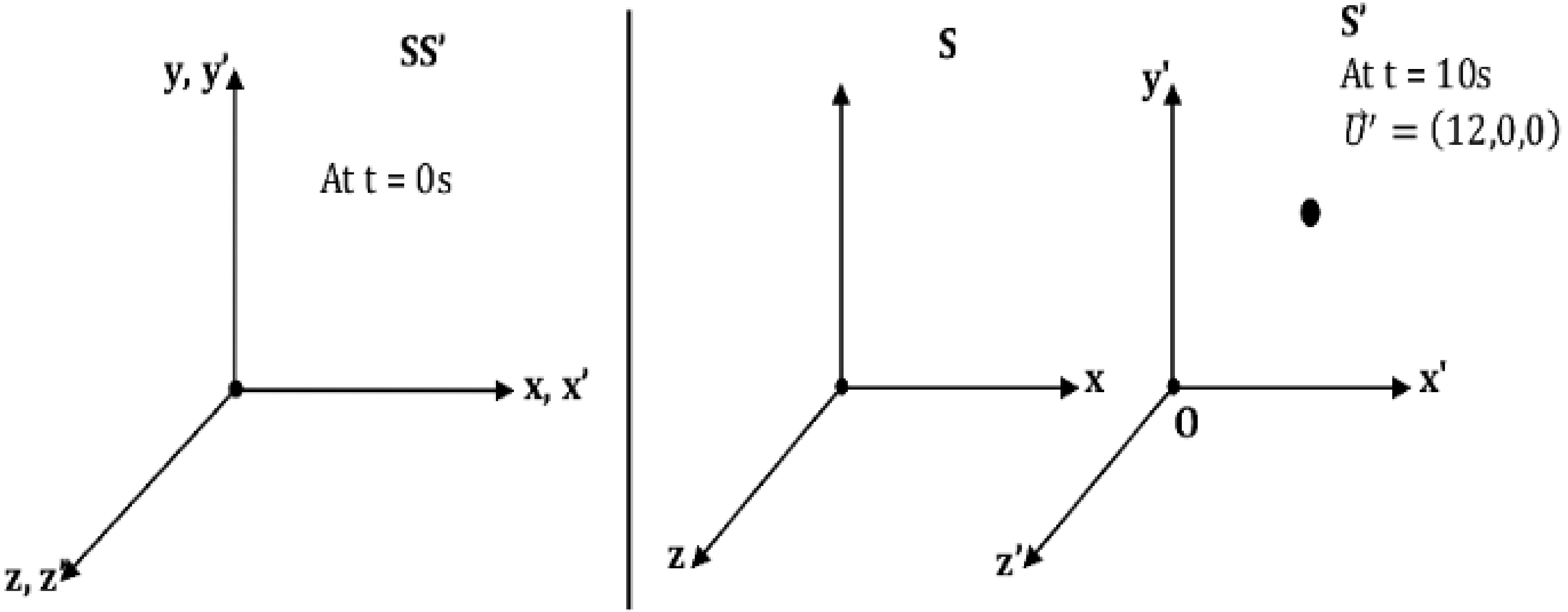
$$x_1 = 50 + 10(120) = 50 + 1200$$

$$x_1 = 1250m$$

Thus, point (50, 0, 0) in frame 2 ≡ point (1250, 0, 0) in frame 1

2: At $t = 0$ seconds, two reference frames consisting of xyz plane coincide such that the origin of one is found at the origin of the second. However, as time progresses, the second reference frame moves in the positive direction along x-axis with velocity of 37m/s, while the first frame of reference remains stationary. At a time of $t = 10$ seconds, a particle within the second reference frame has a velocity given by vector $U'=12, 0, 0$ m/s. Find the equivalent velocity U' of the particle with respect to the first reference frame.

Solution



Galilean velocity transformation equations are:

$$\vec{U}_x = \vec{U}'_x + v, \quad \vec{U}_y = \vec{U}'_y, \quad \vec{U}_z = \vec{U}'_z$$

$$\vec{U} = \left(\vec{U}_x,\, \vec{U}_y,\, \vec{U}_z\right) = \left(\vec{U}'_x + v,\, \vec{U}'_y,\, \vec{U}'_z\right) = (12 + 37, 0, 0)$$

$\therefore\ \vec{U} = (49, 0, 0)\, m/s$ (Velocity of particle w.r.t. reference frame 1)

Universal Gravitation

Newton's law of universal gravitation states that a particle attracts every other particle in the universe with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between their centres. This gravitational force is the same as encountered in Newton's second law of motion, where:

$$F = ma = \frac{GMm}{r^2}$$

Where G is gravitational constant and r is the distance of separation between the masses

Since force, F is Galilean invariant, we can conclude that the law of universal gravitation is also Galilean invariant, supporting the notion that all physical laws are invariant under Galilean transformation.

3: *Show that the law of conservation of linear momentum for collision is Galilean invariant, given that* $m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$

Solution

From Galilean velocity transformation equation,

$$u_1 = u_1' + v$$

$$u_2 = u_2' + v$$

$$v_1 = v_1' + v$$

$$v_2 = v_2' + v$$

Substituting these equations into the equation for momentum conservation, we have

$$m_1(u_1' + v) + m_2(u_2' + v) = m_1(v_1' + v) + m_2(v_2' + v)$$

$$m_1 u_1' + m_1 v + m_2 u_2' + m_2 v = m_1 v_1' + m_1 v + m_2 v_2' + m_2 v$$

Collecting like terms and subtracting, we have

$$m_1 u_1' + m_2 u_2' = m_1 v_1' + m_2 v_2'$$

This shows that the law of conservation of linear momentum is Galilean Invariant i.e. remain the same in different inertial frames of reference.

Exercise

7.1 A girl is riding a bicycle along a straight road at constant speed, and passes a friend standing at a bus stop (event 1). At a time of 60s later, the friend catches a bus (event 2). If the distance separating the events is 126m in the frame of the girl on the bicycle, what is the bicycle's speed? (Ans: 2.10m/s)

7.2 A police car travelling on a straight road at a constant speed of 30m/s catches up with a truck travelling at a constant speed of 20m/s. The police car reaches the truck 25s after it is first sighted. How far ahead of the police car was the truck when it was first sighted? (Ans: 250m)

7.3 Show that the law of conservation of kinetic energy is Galilean Invariant.

$$\text{Hint: }\left(\frac{1}{2}m_1u_1^2+\frac{1}{2}m_2u_2^2=\frac{1}{2}m_2v_1^2+\frac{1}{2}m_2v_2^2\right)$$

Short Multiple-Choice Quiz to Refresh Your Brain

(1) The expression for time dilation is (a) $\Delta t' = \Delta t\sqrt{1 - {v^2}/{c^2}}$ (b) $\Delta t' = \Delta t\sqrt{{v^2}/{c^2} - 1}$ (c) $\Delta t' = \Delta t\sqrt{1 - {c^2}/{v^2}}$ (d) $\Delta t = \Delta t'\sqrt{1 - {v^2}/{c^2}}$

(2) A spaceship leaves Earth on a one-way star trip that earthbound observers judge will take 25 minutes. If the ship travels at 0.95 of speed of light relative to Earth, how long will the trip take as judged by observers in the ship (a) 7.8 minutes (b) 0.78 minutes (c) 1.35 minutes (d) 78 minutes

(3) How long does it take for a meter scale to pass you if it is traveling with a speed of 0.6 of speed of light relative to you along the direction of its length (a) $8.88 \times 10^{-9} s$ (b) $44.4\times10^{-4} s$ (c) $5.63\times10^{-9} s$ (d) $4.44\times10^{-9} s$

(4) Frame of reference is known as (a) measurements of weight and distance in a definite coordinate system (b) measurements of velocity and mass in a definite coordinate system (c) measurements of time and distance in a definite coordinate system (d) a and b

(5) One remarkable consequence of relativity is that simultaneity of events and sometimes even their time order depend on one's ------ (a) time of flight (b) reference frame (c) speed (d) distance covered

(6) Length contraction happens only (a) perpendicular to direction of motion (b) along direction of motion (c) parallel to direction of motion (d) both a and b

(7) Special theory of relativity treats problems involving (a) inertial frame of reference (b) non-inertial frame of reference (c) non-accelerated frame of reference (d) accelerated frame of reference

Answer

(1)a

(2) a

(3)d

(4) c

(5) b

(6) b